A class of algorithms for general instrumental variable models

https://arxiv.org/abs/2006.06366

(NeurIPS 2020)

joint work with

Matt Kusner & **Ricardo Silva**

Niki Kilbertus

HELMHOLTZAI

Motivation

Let’s start with a classic

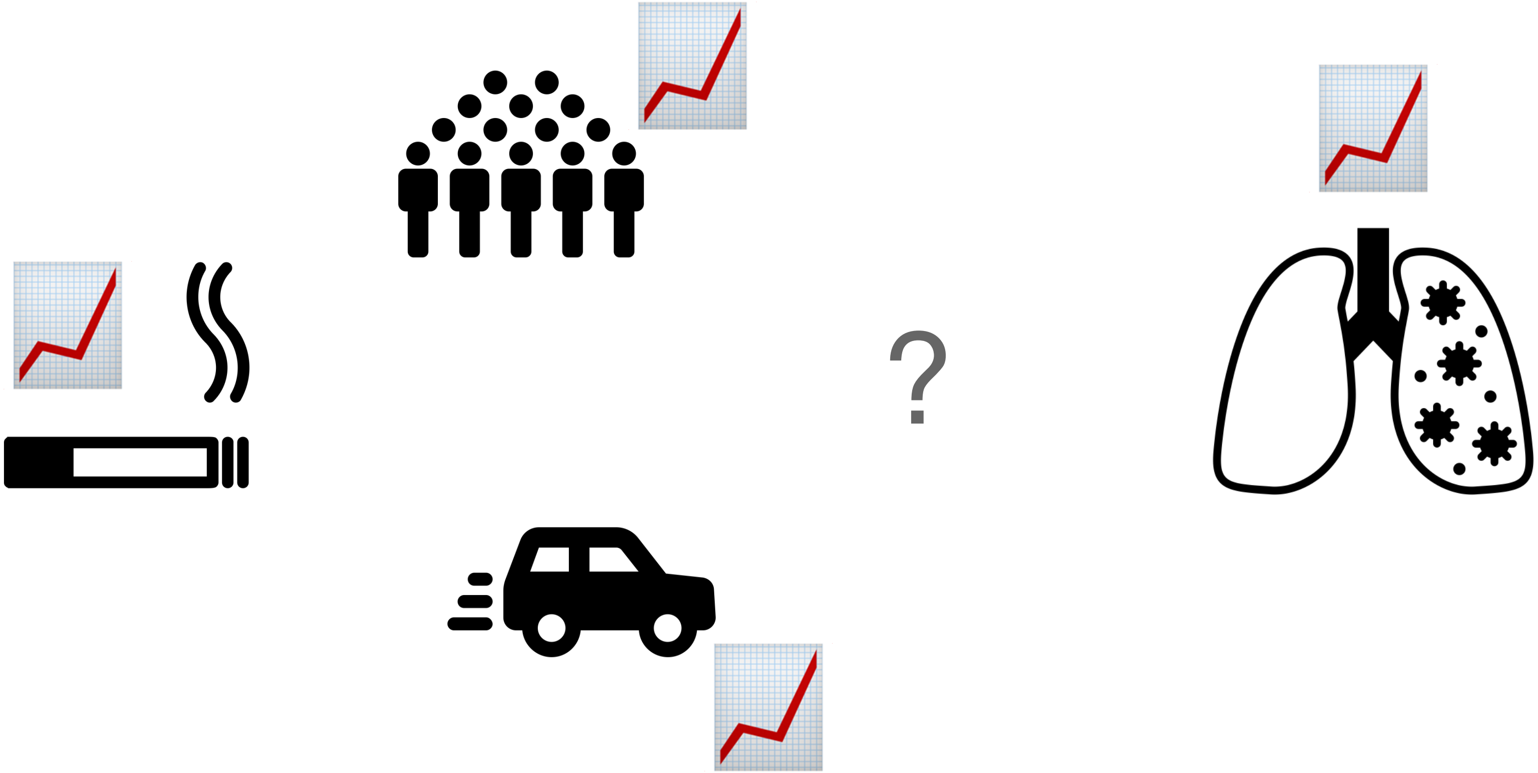


Image credit (from Noun Project):
Andrew Nielsen, mungang kim, Adrien Coquet

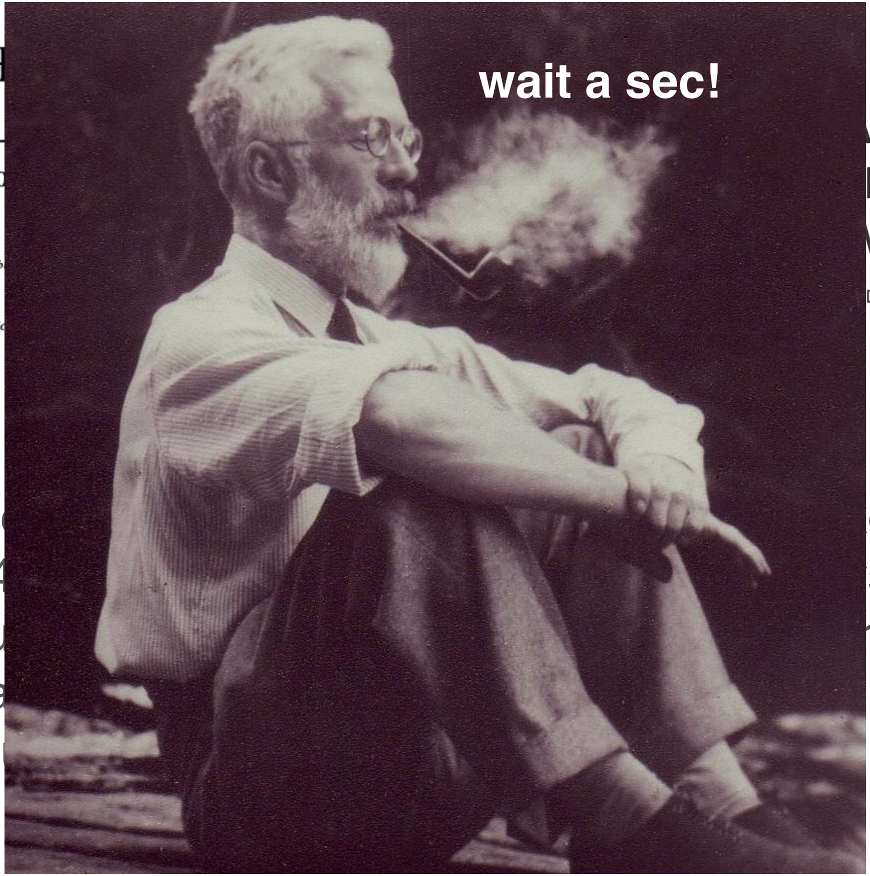
There was “a lot of correlation”

BRITISH

SMO

Memb

Professor of Medical Statistic

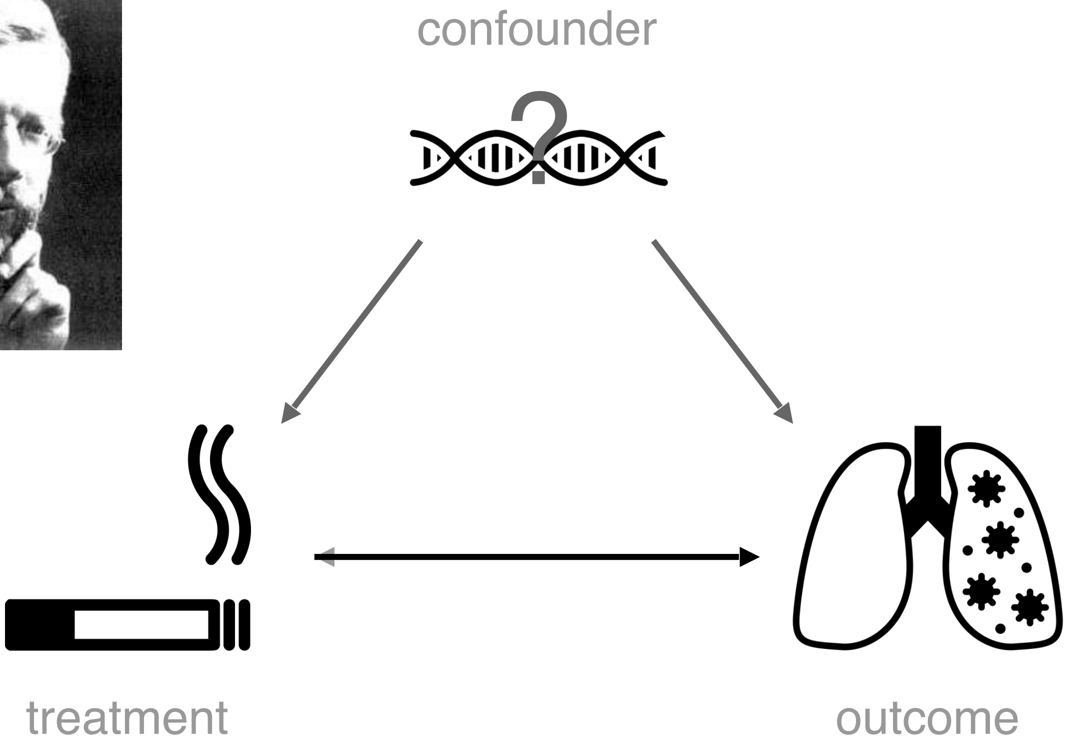


ATIONSHIP BETWEEN HUMAN SMOKING
ND DEATH RATES
W-UP STUDY OF 187,766 MEN

D.; Daniel Horn, Ph.D.

- 3 ers, 56 were heavy smokers
- 14 s. 23.9% other cancer patients
- su rst (all 36 who died of lung
  ca
- m

Unobserved confounding



confounder

?

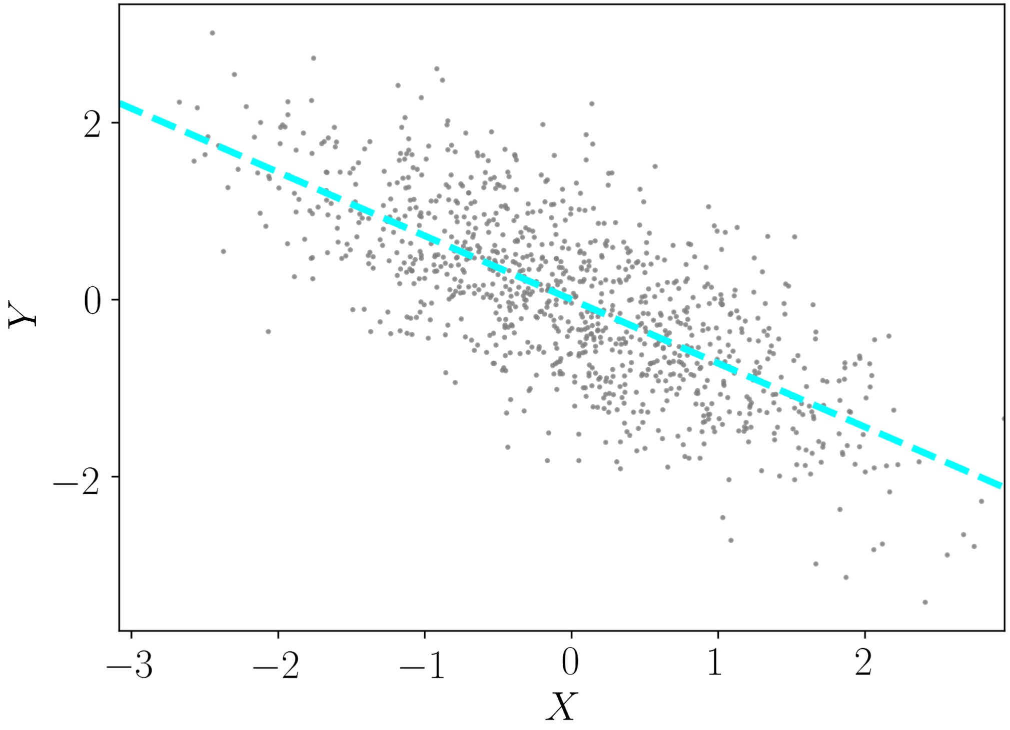
treatment

outcome

Image credit (from Noun Project):
priyanka, Andrew Nielsen, mungang kim

Introduction

Naive ML approach: standard regression



$$X \in \mathbb{R},\ Y \in \mathbb{R}$$

$$\boxed{Y = f(X) + e_Y}$$

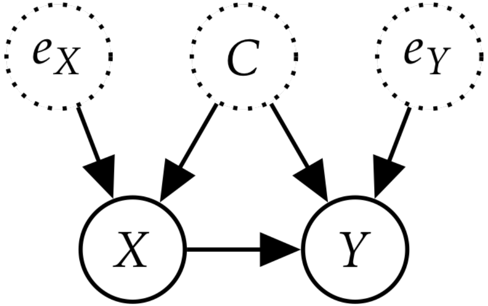
$$\boxed{\mathbb{E}[e_Y \mid X] = 0}$$

$$\mathbb{E}[Y - f(X) \mid X] = 0$$

$$\Rightarrow \mathbb{E}[Y \mid X] = f(X)$$

linear least squares: $f = \arg\min_{\hat{f}} \sum_i \left(\hat{f}(x_i) - y_i\right)^2$

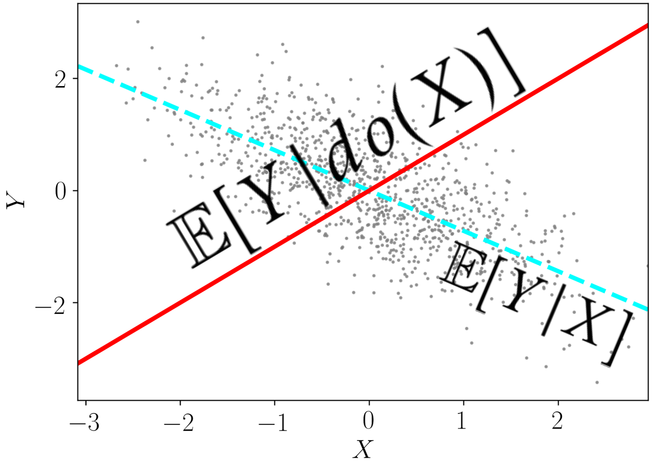
Naive ML approach failing



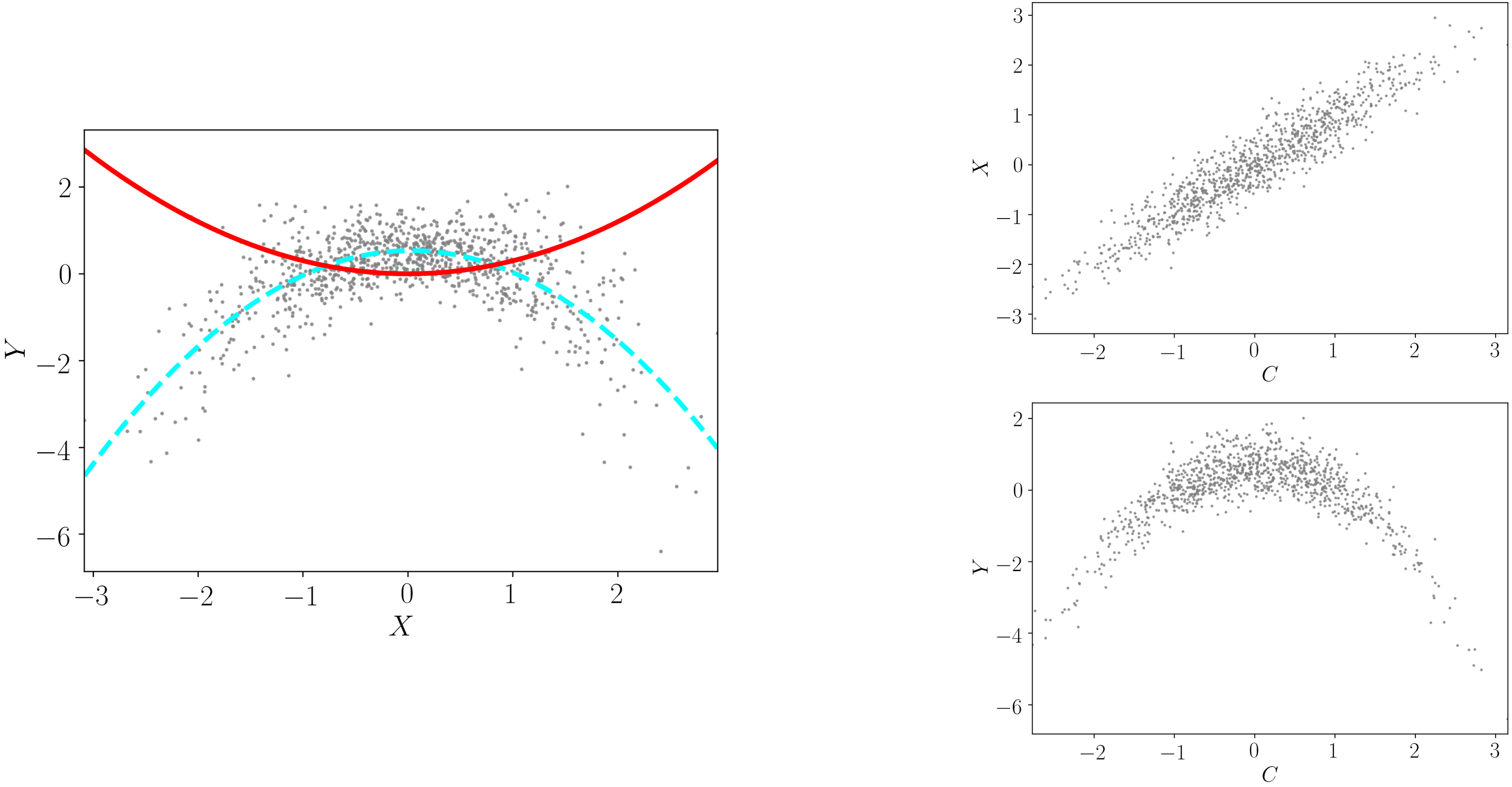
$$X = \alpha \cdot C + e_X$$

$$Y = \boxed{X} + \boxed{\beta \cdot C + e_Y} \longleftrightarrow Y = \boxed{f(X)} + \boxed{e_Y}$$

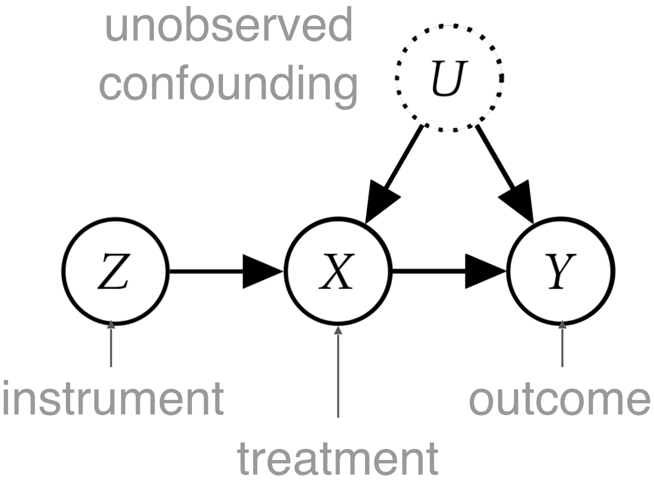
$$\xcancel{\mathbb{E}[e_Y | X] = 0}$$



Losing hope...



Instrumental variables



(a) Z influences X $\quad Z \not\perp\!\!\!\perp X$

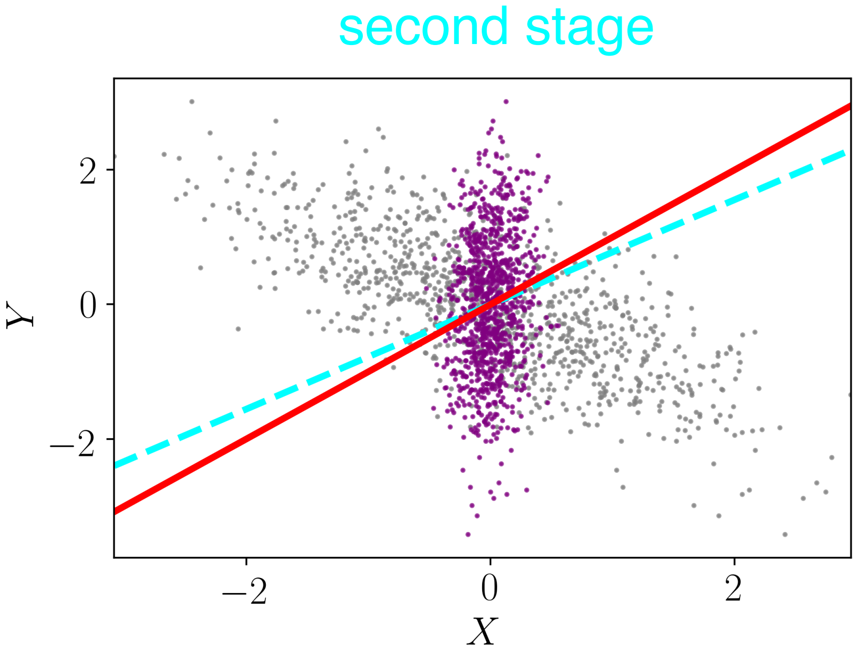
(b) Z is independent of U $\quad Z \perp\!\!\!\perp U$

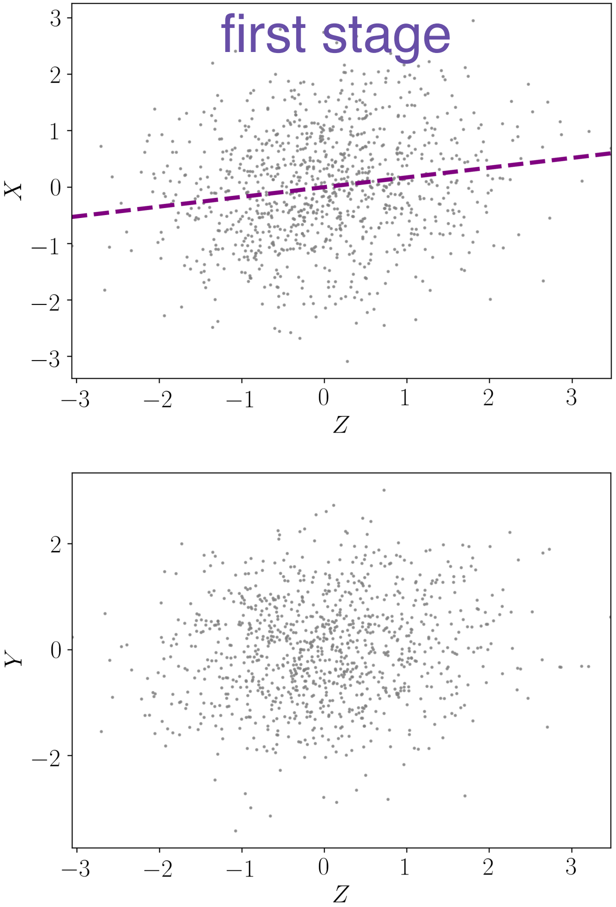
(c) Z only influences Y via X $\quad Z \perp\!\!\!\perp Y \,|\, \{X, U\}$

assume: $Y = f(X) + e_Y \quad \text{with} \quad \mathbb{E}[e_Y] = 0$

$$\underbrace{\boxed{\mathbb{E}[Y|z]}}_{\text{identifiable}} = \mathbb{E}[f(X) + e_Y|z] = \mathbb{E}[f(X)|z] = \int \underbrace{\boxed{f(x)}}_{\text{unique under mild conditions}} \underbrace{\boxed{p(x|z)}}_{\text{identifiable}} dx$$

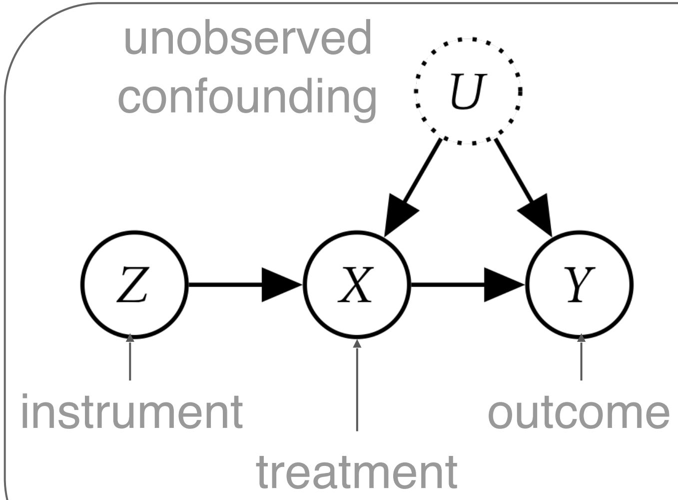
Two stage least squares (2SLS) -- linear case





Problem formulation

General problem formulation



Assumptions

(a) Z influences X $Z \not\perp\!\!\!\perp X$

(b) Z is independent of U $Z \perp\!\!\!\perp U$

(c) Z only influences Y via X $Z \perp\!\!\!\perp Y \mid \{X, U\}$

$$X = g(Z, U) \qquad Y = f(X, U)$$

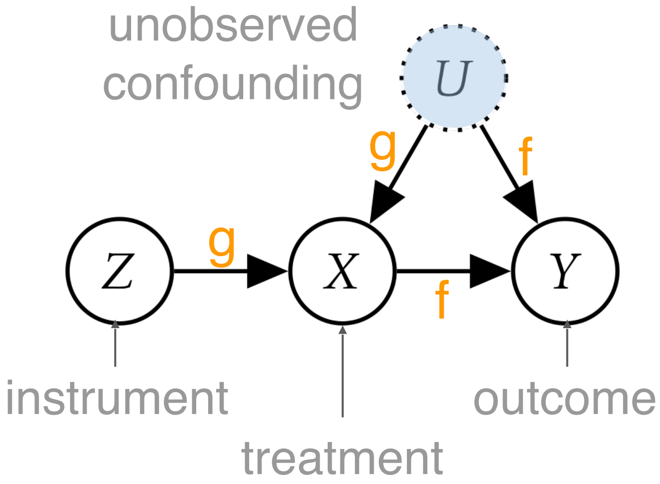
non-linear, non-additive

Goal - partial identification

For any x^* compute lower and upper bounds on the causal effect

$$\mathbb{E}[Y \mid do(x^\star)]$$

General problem formulation as optimization



optimize over "all" distributions

$$X = g(Z, U) \qquad Y = f(X, U)$$

optimize over "all" functions

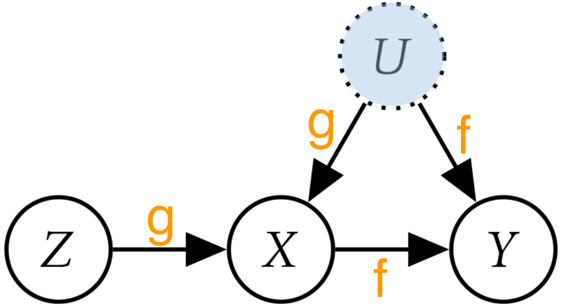
Goal

among all possible {*g*, *f*} and distributions over *U*
that reproduce the observed densities {p(x | z), p(y | z)},
estimate the min and max expected outcomes under intervention

Operationalizing this optimization

- without any restrictions on functions and distributions:
  effect is not identifiable and average treatment effect bounds are vacuous
  [Pearl, 1995; Bonet, 2001; Gunsilius 2018]
- mild assumptions suffice for meaningful bounds:
  f and *g* have a finite number of discontinuities [Gunsilius, 2019]
- rest of the talk: **operationalize the optimization**

find convenient representation of U from which we can sample

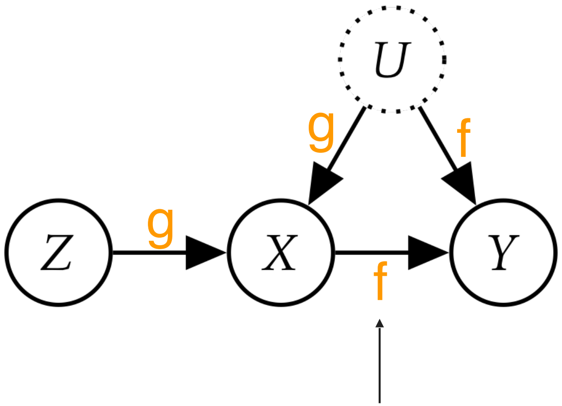


choose convenient function spaces

approximate constraints of preserving p(x I z) and p(y I z)

Our practical approach

Response functions I [Balke & Pearl, 1994]



ultimately, we care about this functional relation

- each value of *U* fixes a functional relation *X* → *Y*
- collect the set of all resulting functions $\{f_u\}$
- identify values of *u* that result in the same f_u and assign a unique index *r*

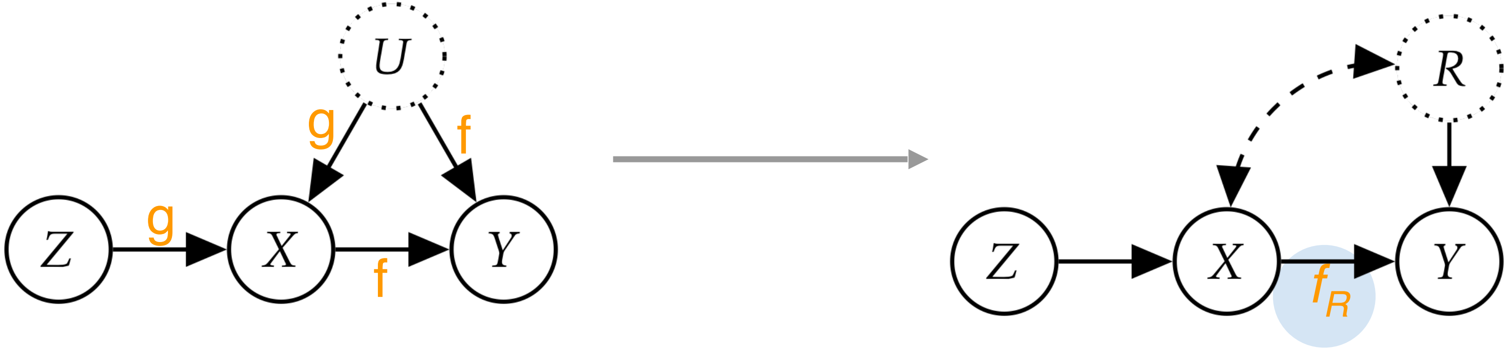
$$Y = f(X, U) = \lambda_1 X + \lambda_2 X U_1 + U_2$$

$$f(x, u) = \lambda_1 x + \lambda_2 x \quad \text{for} \quad u_1 = 1, u_2 = 0$$

$$f_r(x) = (\lambda_1 + \lambda_2) x \quad \text{where} \quad r \text{ is an alias for } (1, 0)$$

→ Instead of a potentially multivariate distribution over confounders *U* directly, we can think of a distribution *R* over functions *f: X* → *Y*

Response functions II



choose convenient function spaces

find convenient representation of U from which we can sample

find convenient representation of distributions over response functions

Parameterizing response functions

We choose a simple parameterization

$$f_r(x) := f_{\theta_r}(x) \quad \text{for} \quad \theta \in \Theta \subset \mathbb{R}^K$$

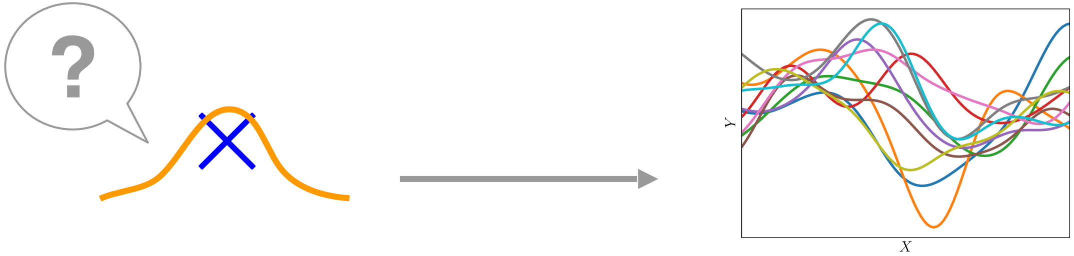
For simplicity, work with linear combination of (non-linear) basis functions:

$$f_\theta(x) = \sum_{k=1}^{K} \theta_k \psi_k(x) \quad \text{for basis functions} \quad \{\psi_k : \mathbb{R} \to \mathbb{R}\}_{k=1}^{K}$$

polynomials

neural networks

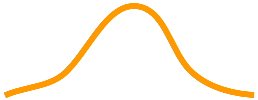
Gaussian process samples



$$\theta$$

$$f_\theta : X \to Y$$

Parameterizing the distribution over θ



$p_{\mathcal{M}}(\theta)$ implies a causal model

Goal

optimize over distributions $p_{\mathcal{M}}(\theta)$ such that

$$\int p_{\mathcal{M}}(x, y | z, \theta) p_{\mathcal{M}}(\theta) \, d\theta \quad \text{matches (estimated) marginals} \quad p(x|z), p(y|z)$$

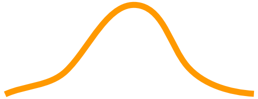
ideally

low variance Monte-Carlo gradient estimation

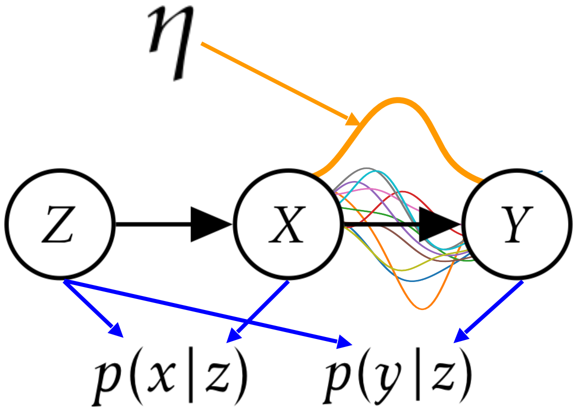
differentiable sampling

again, assume parametric form of $p_{\mathcal{M}}(\theta)$

$$p_\eta(\theta) \quad \text{with} \quad \eta \in \mathbb{R}^d$$



Objective function

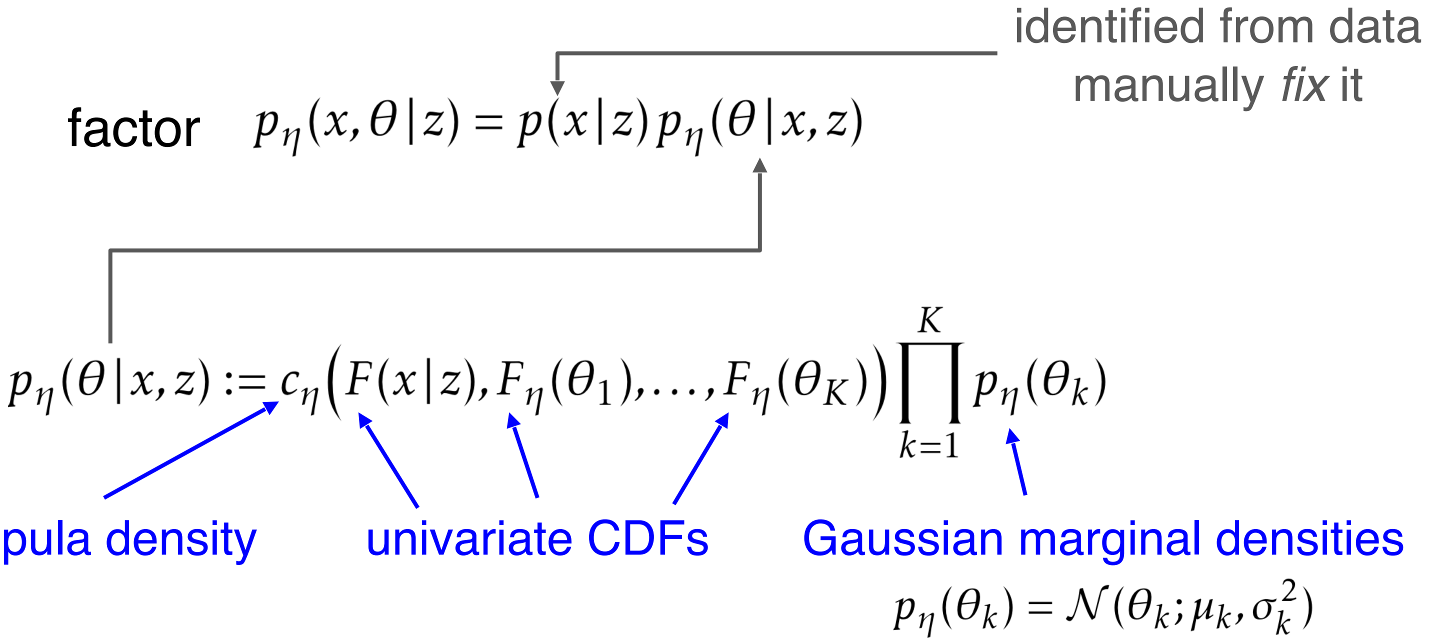


objective

$$\min_\eta / \max_\eta \mathbb{E}[Y \,|\, do(x^\star)] = \min_\eta / \max_\eta \int f_\theta(x^\star) p_\eta(\theta)\, d\theta$$

Our model must match the observed data. Next up: Add these constraints.

Match p(x l z) and enforcing Z ⊥ U



identified from data
manually *fix* it

factor $p_\eta(x,\theta|z) = p(x|z)\,p_\eta(\theta|x,z)$

$$p_\eta(\theta|x,z) := c_\eta\Big(F(x|z), F_\eta(\theta_1),\ldots,F_\eta(\theta_K)\Big)\prod_{k=1}^{K} p_\eta(\theta_k)$$

copula density

univariate CDFs

Gaussian marginal densities

$$p_\eta(\theta_k) = \mathcal{N}(\theta_k;\mu_k,\sigma_k^2)$$

for a multivariate Gaussian copula, the optimization parameters are

$$\eta := \{\mu_1, \ln(\sigma_1^2),\ldots,\mu_K,\ln(\sigma_K^2), L\} \in \mathbb{R}^{K(K+1)/2+2K}$$

Match p(y | z)

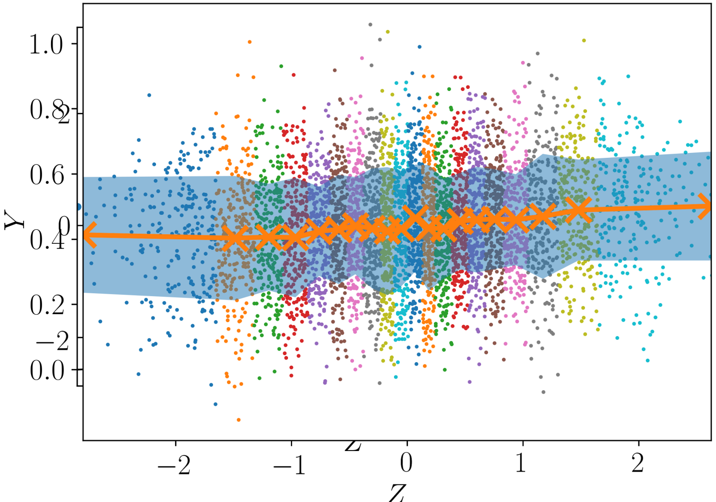
exact constraint in the continuous outcome setting

data $$\Pr(Y \le y \,|\, Z = z) = \int \mathbf{1}(f_\theta(x) \le y)\, p_\eta(x, \theta \,|\, z)\, dx\, d\theta$$ our model

- infinite number of constraints
- integral over non-continuous indicator

choose discrete finite grid in z and assign points to bins

$$z^{(m)} := F_Z^{-1}\left(\frac{m}{M+1}\right) \quad \text{for } m \in [M]$$



for a dictionary of basis functions $\{\phi_l\}_{l=1}^{L}$

$$\mathbb{E}[\phi_l(Y) \,|\, z^{(m)}] = \int \phi_l\big(f_\theta(x)\big)\, p_\eta(x, \theta \,|\, z^{(m)})\, dx\, d\theta$$

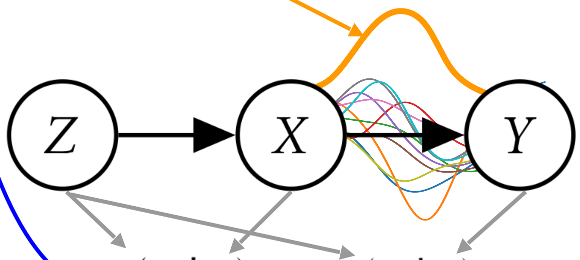
data our model

$$\phi_1(Y) := \mathbb{E}[Y],\ \phi_2(Y) := \mathbb{V}[Y]$$

Intermediate overview

$Z \perp\!\!\!\perp \theta$

copula: η



fix manually

$p(x|z)$ $p(y|z)$

explicit constraint

$$\mathbb{E}[\phi_l(Y)|z^{(m)}] = \int \phi_l\big(f_\theta(x)\big)p_\eta(x,\theta|z^{(m)})\,dx\,d\theta$$

objective

$$\min_\eta/\max_\eta \mathbb{E}[Y|do(x^\star)] = \min_\eta/\max_\eta \int f_\theta(x^\star)p_\eta(\theta)\,d\theta$$

The final optimization problem

can sample from these in a differentiable fashion (w.r.t. η)

objective: $$o_{x^\star}(\eta) := \int f_\theta(x^\star)\boxed{p_\eta(\theta)}\,d\theta$$

precompute once up front from data

constraint LHS: $$\mathrm{LHS}_{m,l} := \mathbb{E}[\phi_l(Y)|z^{(m)}]$$

constraint RHS: $$\mathrm{RHS}_{m,l}(\eta) := \int \phi_l(f_\theta(x))\boxed{p_\eta(x,\theta|z^{(m)})}\,dx\,d\theta$$

opt. problem: $$\min_\eta/\max_\eta o_{x^\star}(\eta) \quad \text{s.t.} \quad \boxed{\mathrm{LHS}_{m,l}} = \mathrm{RHS}_{m,l}(\eta) \text{ for all } m \in [M], l \in [L]$$

only satisfy this approximately

use augmented Lagrangian with stochastic gradient descent

- for each $z^{(m)}$ sample batch of θ
- take average to estimate objective and constraint term RHS
- use auto-differentiation and gradient-based optimization

Empirical results

Choices of response functions

$$f_\theta(x) = \sum_{k=1}^{K} \theta_k \psi_k(x) \quad \text{for basis functions} \quad \{\psi_k : \mathbb{R} \to \mathbb{R}\}_{k=1}^{K}$$

Polynomials

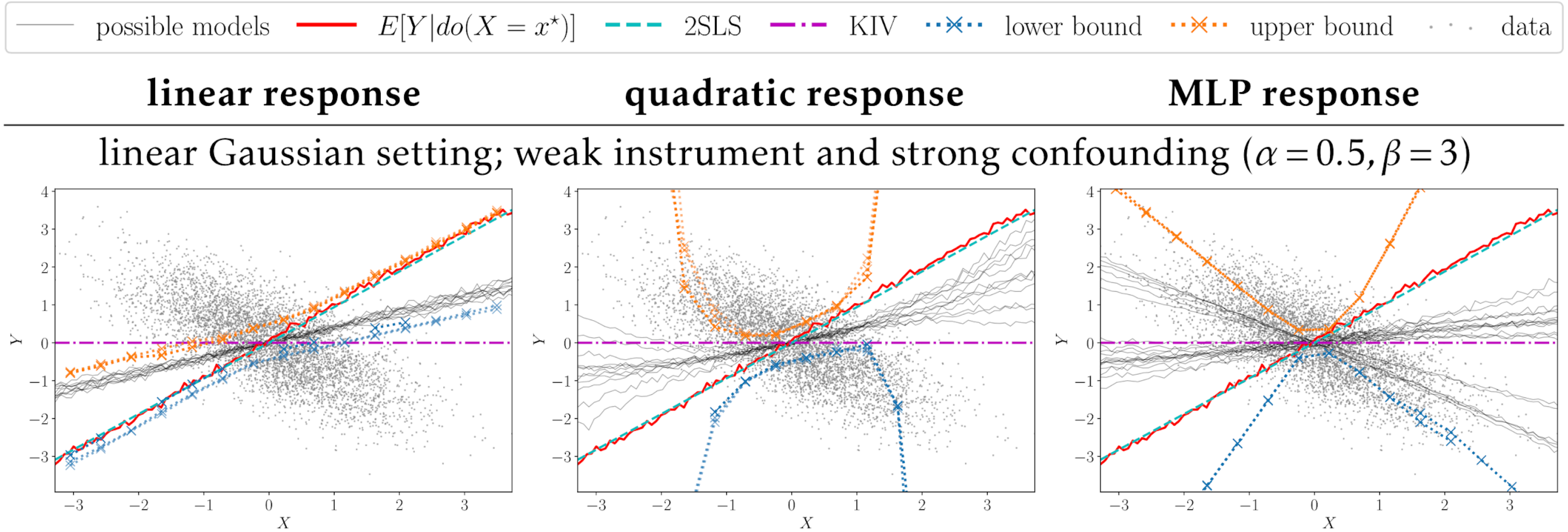
$\psi_k(x) = x^{k-1}$ for $k \in [K]$

Neural network

Train a small fully connected network on observed data $X \to Y$ and take activations of last hidden layer as basis functions.

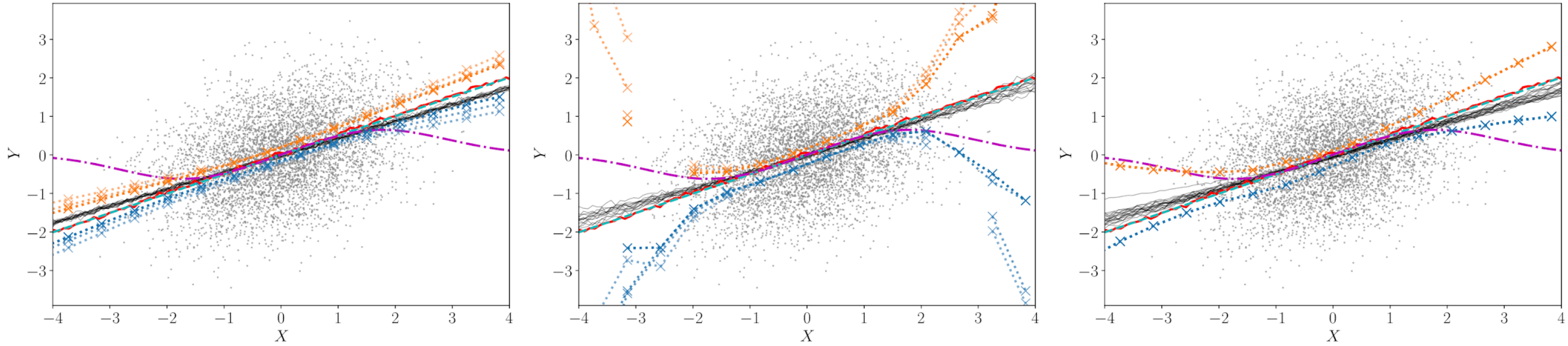
Gaussian process

Train GPs on subsets of observed data X→Y and take random samples from the GP as basis functions.



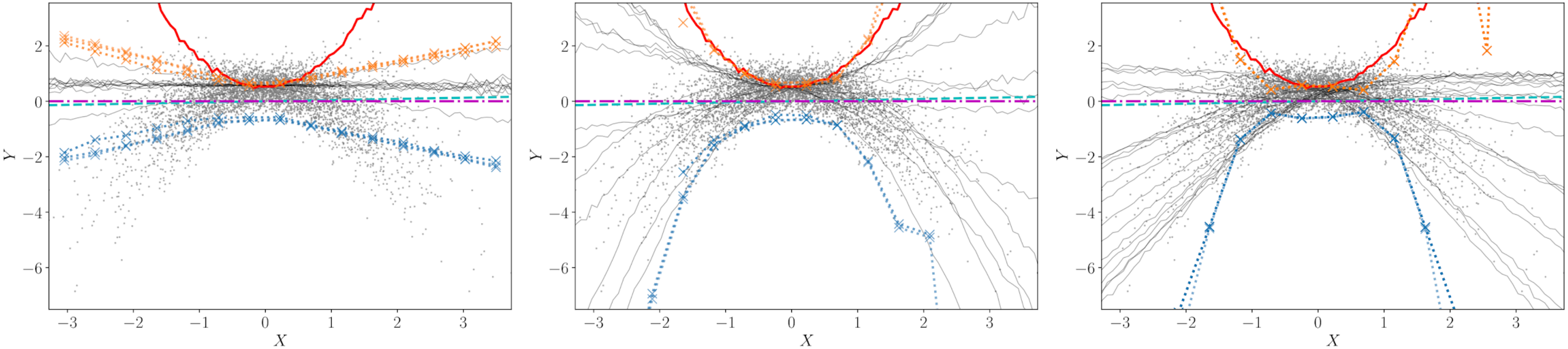
linear response **quadratic response** **MLP response**

linear Gaussian setting; weak instrument and strong confounding ($\alpha = 0.5, \beta = 3$)

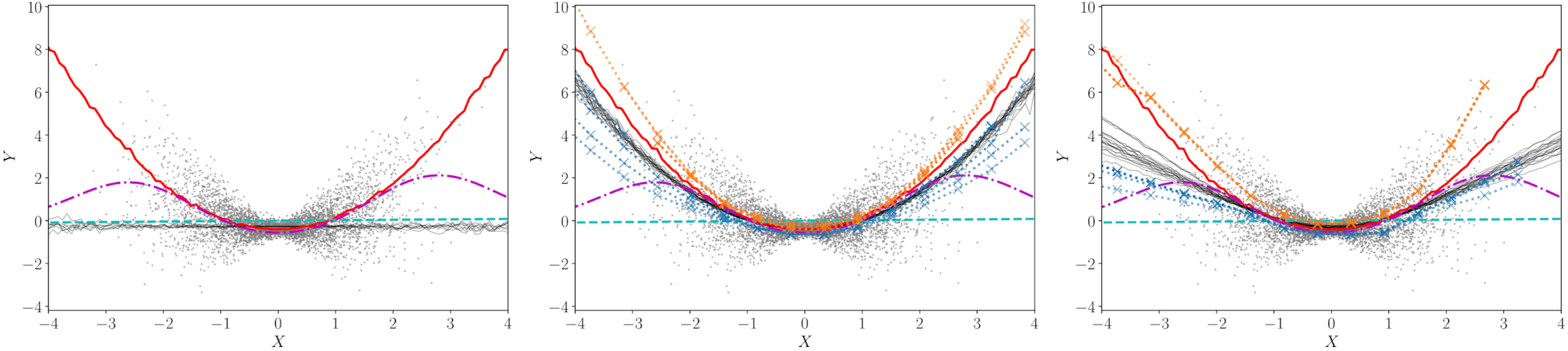


linear Gaussian setting; strong instrument and weak confounding ($\alpha = 3, \beta = 0.5$)

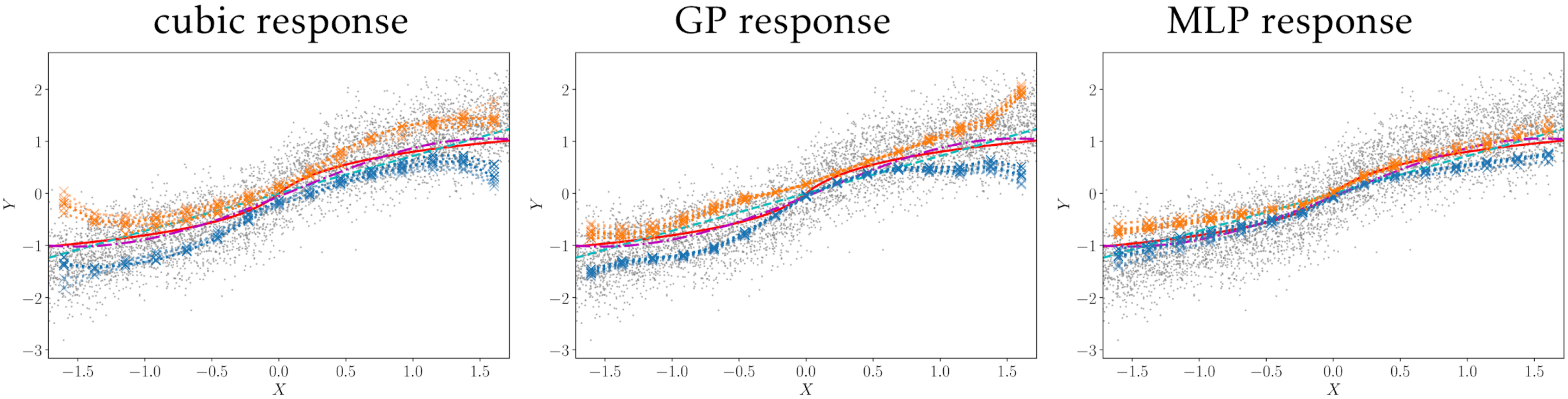
non-additive, non-linear setting; weak instrument and strong confounding ($\alpha = 0.5, \beta = 3$)



non-additive, non-linear setting; strong instrument and weak confounding ($\alpha = 3, \beta = 0.5$)



Sigmoidal cause-effect design



more details and experiments (also in the small data regime) in the paper
https://arxiv.org/abs/2006.06366

Thank you